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Uncertainty relations in deformation quantization*

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Abstract
Robertson and Hadamard–Robertson theorems on non-negative definite Hermitian forms are generalized to an arbitrary ordered field. These results are then applied to the case of formal power series fields, and the Heisenberg–Robertson, Robertson–Schrödinger and trace uncertainty relations in deformation quantization are found. Some conditions under which the uncertainty relations are minimized are also given.

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Dedicated to Jerzy Plebański on the occasion of his 75th birthday.

1. Introduction

The Heisenberg uncertainty relation for canonical observables q and p is certainly one of the most fundamental results in quantum mechanics. It was introduced by Heisenberg in 1927 [1] and mathematically proved by Kennard [2] and Weyl [3]. Later on the Heisenberg uncertainty relation was generalized to the case of two arbitrary observables by Robertson [4, 5] and Schrödinger [6]. In fact in [5, 6] an improved version of the Heisenberg uncertainty relation has been obtained. Finally, Robertson [7] was able to extend the previous results to an arbitrary number of observables. The inequalities found in [7] are called the *Heisenberg–Robertson* and *Robertson–Schrödinger uncertainty relations*.

Recently a great deal of interest in uncertainty relations has been observed. It has been shown that they can be used to define *squeezed* and *coherent states* and also to generalize these important concepts by introducing the notion of *intelligent states* [8–18].

* We would like to dedicate this modest work to our teacher and friend, Professor Jerzy Plebański, who several years ago showed us his works on Moyal bracket and the beautiful notes from his Polish lectures entitled 'Nawiasy Poissona i Komutatory' [30]. This was the inspiration of our interest in deformation quantization.

It seems natural that any theory which would like to describe quantum systems should reproduce in some sense the uncertainty relations. So we expect that this must also be the case in deformation quantization.

Deformation quantization as introduced by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [19] and extensively developed during recent years (for a review see [20, 21]), besides a well-constructed mathematical formalism is expected to be an alternative approach to the description of quantum systems. A big effort in this direction has been made, e.g. [22–28].

The aim of the present paper is to study the uncertainty relations in deformation quantization. This problem in the case of two observables has already been considered by Curtright and Zachos [29]. We are going to extend their results to the case of an arbitrary number of observables (real formal power series) and so to obtain in deformation quantization the Heisenberg–Robertson and Robertson–Schrödinger uncertainty relations and also the concept of Robertson–Schrödinger intelligent state.

To deal with uncertainty relations in deformation quantization first one should consider the theory of formally real ordered fields and then apply it to the field of formal power series. This is done in sections 2 and 3.

The importance of the theory of formally real ordered fields in deformation quantization and especially in the Gel'fand–Naimark–Segal (GNS) construction was recognized by Bordemann and Waldmann [25]. In our paper we use extensively the results of their distinguished work.

In section 4 the proofs of Robertson and Hadamard–Robertson theorems for an arbitrary ordered field are given. The results of this section are then used in section 5 to obtain the Heisenberg–Robertson, Robertson–Schrödinger and trace uncertainty relations. Some conditions to minimize the Robertson–Schrödinger uncertainty relations and to get intelligent states are found in section 6. These conditions are the deformation quantization analogues of those introduced by Trifonov [12].

Finally, some concluding remarks in section 7 close our paper.

2. Formally real fields

In this section we give a brief review of the Artin–Schreier theory of formally real fields. For a detailed exposition the reader is referred to the books by Jacobson [31], Lang [32], Fuchs [33], Rajwade [34], Scharlau [35], Prestel and Delzell [36] or to the original paper by Artin and Schreier [37].

Let $\mathbb{K}$ be a field.

Definition 2.1. *An ordered field is a pair* $(\mathbb{K}, P)$ *where* P *is a subset of* $\mathbb{K}$ *such that*

(i) $0 \notin P$, $P \cap -P = \emptyset$;
(ii) $P + P \subset P$, $P \cdot P \subset P$;
(iii) $\mathbb{K} = P \cup \{0\} \cup -P$.

If $(\mathbb{K}, P)$ is an ordered field then we say that $\mathbb{K}$ *is ordered by* P and P is called an *order of* $\mathbb{K}$ or the *set of positive elements of* $\mathbb{K}$. It is easy to show that if P and P' are two orders of $\mathbb{K}$ and $P' \subset P$ then $P' = P$.

Let $a \neq 0$ be any element of $\mathbb{K}$. By (iii) $a \in P$ or $-a \in P$. Then by (ii) $a^2 = (-a)^2 \in P$. Consequently, if $a_i \in \mathbb{K}, i = 1, \ldots, n$, then $a_1^2 + \cdots + a_n^2 = 0$ iff $a_i = 0$ $\forall i$. Now, since $1 = 1^2 \in P$ one has $1 + \cdots + 1 \neq 0$ which means that the characteristic of $\mathbb{K}$ is 0.

One defines the relations $>$ and $\geqslant$ by: $a > b$ for $a, b \in \mathbb{K}$ iff $a - b \in P$; $a \geqslant b$ iff $a > b$ or $a = b$. The following properties of the relation $>$ can be easily proved:

$$a > 0 \quad \text{iff} \quad a \in P$$

$$a > b \quad \text{and} \quad b > c \quad \Rightarrow \quad a > c$$

$$a \neq b \quad \Rightarrow \quad a > b \quad \text{or} \quad b > a$$

$$a > b \quad \Rightarrow \quad a + c > b + c \quad \text{for any} \quad c \in \mathbb{K}$$

$$a > b \quad \Rightarrow \quad ad > bd \quad \text{for any} \quad d \in P.$$

As is used in real number theory we write $b \leqslant a$ iff $a \geqslant b$.

Given an ordered field $(\mathbb{K}, P)$ the module $|\cdot|$ can be defined by: $|a| = a$ for $a > 0$, $|a| = -a$ for $a \leqslant 0$. One quickly finds that $|ab| = |a||b|$ and $|a + b| \leqslant |a| + |b|$.

Definition 2.2. $(\mathbb{K}, P)$ *is called an Archimedean ordered field if for each* $a \in \mathbb{K}$ *there exists an* $n \in \mathbb{N}$ *such that* $1 + \cdots + 1 > a$.

An important class of fields called *formally real fields* was introduced and analysed by Artin and Schreier in their pioneer work [37].

Definition 2.3. $\mathbb{K}$ *is said to be a formally real field if* -1 *is not a sum of squares in* $\mathbb{K}$.

The classical example of this type of field is provided by the real number field $\mathbb{R}$. Another example fundamental for our further constructions will be given in the next section.

The connection between the ordered fields and the formally real fields is given by

Theorem 2.1. $\mathbb{K}$ *can be ordered iff* $\mathbb{K}$ *is formally real.*

Definition 2.4. *A field* $\mathbb{K}$ *is called real closed if*

(i) $\mathbb{K}$ *is formally real;*
(ii) any formally real algebraic extension of $\mathbb{K}$ *is equal to* $\mathbb{K}$.

For example, the real number field $\mathbb{R}$ is real closed.

The following theorems characterize the real closed fields:

Theorem 2.2. *If* $\mathbb{K}$ *is real closed, then* $\mathbb{K}$ *has a unique order* $P = (\mathbb{K} - \{0\})^2 := \{a^2 : a \in \mathbb{K} - \{0\}\}$.

Theorem 2.3. *The following statements are equivalent:*

(1) $\mathbb{K}$ *is real closed.*
(2) Any polynomial of odd degree with coefficients in $\mathbb{K}$ *has a root in* $\mathbb{K}$ *and there exists an order P of* $\mathbb{K}$ *such that any positive element has square root in* $\mathbb{K}$.
(3) $\sqrt{-1} \notin \mathbb{K}$ *and* $\mathbb{K}(\sqrt{-1})$ *is algebraically closed.*

(We use the usual notation in which $\mathbb{K}(X_1, \ldots, X_n)$ denotes the field of rational functions in $X_1, \ldots X_n$ with coefficients in $\mathbb{K}$. So $\mathbb{K}(\sqrt{-1}) = \mathbb{K} + \sqrt{-1}\mathbb{K}$.)

Now the natural question arises if an arbitrary ordered field can be extended to a real closed one. To answer this question first we give

Definition 2.5. *Let* $(\mathbb{K}, P)$ *be an ordered field. A field* $\mathbb{K}'$ *is said to be a real closure of* $\mathbb{K}$ *relative to P if the following conditions are satisfied:*

(i) $\mathbb{K}'$ *is an algebraic extension of* $\mathbb{K}$;

(ii) $\mathbb{K}'$ *is real closed;*
(iii) $P = (\mathbb{K}' - \{0\})^2 \cap \mathbb{K}$, *i.e., the unique order* $(\mathbb{K}' - \{0\})^2$ *of* $\mathbb{K}'$ *is an extension of* P.

Perhaps the most important result in the formally real fields theory is the theorem due to Artin and Schreier [37] on the existence and uniqueness of a real closure for any ordered field.

Theorem 2.4. *Any ordered field* $(\mathbb{K}, P)$ *has a real closure relative to* P. *If* $(\mathbb{K}_1, P_1)$ *and* $(\mathbb{K}_2, P_2)$ *are ordered fields and* $\mathbb{K}_1'$ *and* $\mathbb{K}_2'$ *their respective closures, then any isomorphism* $f : \mathbb{K}_1 \to \mathbb{K}_2$ *such that* $f(P_1) = P_2$ *can be uniquely extended to an isomorphism* $f' : \mathbb{K}_1' \to \mathbb{K}_2'$ *with* $f'((\mathbb{K}_1' - \{0\})^2) = (\mathbb{K}_2' - \{0\})^2$.

This theorem will be applied in section 4 to prove the generalized Robertson inequality.

Finally, we introduce the notion of an exponential valuation of an arbitrary field $\mathbb{K}$.

Definition 2.6. *Let* $\mathbb{K}$ *be a field. An exponential valuation of* $\mathbb{K}$ *is a mapping* $\nu : \mathbb{K} \to \mathbb{R} \cup \{\infty\}$ *such that for all* $a, b \in \mathbb{K}$

(i) $\nu(a) = \infty \Leftrightarrow a = 0$
(ii) $\nu(ab) = \nu(a) + \nu(b)$
(iii) $\nu(a + b) \geqslant \min\{\nu(a), \nu(b)\}$.

Given an exponential valuation $\nu : \mathbb{K} \to \mathbb{R} \cup \{\infty\}$ one can define a metric on $\mathbb{K}$ as follows:

$$d_\nu(a, b) := \exp\{-\nu(a - b)\} \qquad (\exp\{-\infty\} := 0). \tag{2.1}$$

The pair $(\mathbb{K}, d_\nu)$ is a metric space, and consequently all notions known in the theory of metric spaces can be applied in the present case e.g., a topology $\mathcal{T}_\nu$ defined by the metric d_ν, Cauchy sequences, completeness, etc.

Let $(\mathbb{K}, P)$ be an ordered field. Then we have a natural topology $\mathcal{T}_0$ on $\mathbb{K}$: a base of $\mathcal{T}_0$ is the set $\mathcal{B}$ of ε-balls, where the *ε-ball with centre at $a \in \mathbb{K}$*, $B_\varepsilon(a)$, is defined by

$$B_\varepsilon(a) := \{b \in \mathbb{K} : |b - a| < \varepsilon\} \qquad 0 < \varepsilon \in \mathbb{K}.$$

If the topology $\mathcal{T}_\nu$ on $\mathbb{K}$ defined by the valuation $\nu : \mathbb{K} \to \mathbb{R} \cup \{\infty\}$ is equal to the topology $\mathcal{T}_0$ defined by the order P of $\mathbb{K}$ then we say that the *valuation ν is compatible with the ordering of* $\mathbb{K}$.

An important example when $\mathcal{T}_\nu = \mathcal{T}_0$ is considered in the next section.

3. Fields of formal power series

Formal power series play an important role in mathematical physics. Examples of this are the formal solution of the evolution Schrödinger equation or the Baker–Campbell–Hausdorff formula [38]. But maybe the most transparent application of formal power series theory can be found in deformation quantization as formulated by Bayen *et al* [19] and developed by Fedosov [39], Kontsevich [40] and others (see [20, 21]). Here the formal power series with respect to the deformation parameter $\hbar$ arise as the main objects of the construction.

We give a short exposition of general formal power series field theory. For details see [25, 26, 31, 33, 36, 41, 42].

Let $(G, +)$ be an additive Abelian group.

Definition 3.1. *An ordered Abelian group is a pair* $((G, +), S)$ *where* S *is a subset of* G *such that*

(i) $0 \notin S$, $S \cap -S = \emptyset$;

(ii) $S + S \subset S$;
(iii) $G = S \cup \{0\} \cup -S$.

We use the symbol 0 for the neutral element of the group $(G, +)$ as well as for the zero element of a field.

If $g_1, g_2 \in G$ then we say that $g_1 < g_2$ (g_1 *is less than* g_2) iff $g_1 - g_2 \in S$. So $g \in S$ iff $g < 0$ and it means that S consists of elements of G less than the neutral element 0. We write $g_1 \leqslant g_2$ iff $g_1 < g_2$ or $g_1 = g_2$.

Definition 3.2. *Let* $((G, +), S)$ *be an ordered Abelian group and* $\mathbb{K}$ *a field. A formal power series on G over* $\mathbb{K}$ *is a map* $a : G \to \mathbb{K}$ *such that any nonempty subset of the set* $\operatorname{supp} a := \{g \in G : a(g) \neq 0\}$ *has a least element.*

The formal power series $a : G \to \mathbb{K}$ is usually denoted by $a = \sum_{g\in G} a_g t^g$ where $a_g := a(g)$. The set of all formal power series on G over $\mathbb{K}$ will be denoted by $\mathbb{K}((t^G))$. When $(G, +)$ is the Abelian group of integers $(\mathbb{Z}, +)$ we simply write $\mathbb{K}((t))$.

Addition and multiplication of formal power series $a = \sum_{g\in G} a_g t^g$ and $b = \sum_{g\in G} b_g t^g$ are defined as follows:

$$a + b = \sum_{g\in G} (a_g + b_g) t^g \qquad ab = \sum_{g\in G} \left(\sum_{g_1\in G} a_{g_1} b_{g-g_1} \right) t^g. \tag{3.1}$$

(Note that according to definition 3.2 both operations are well defined. In particular for any $g \in G$ the number of nonzero elements of the form $a_{g_1} b_{g-g_1}$, $g_1 \in G$, is finite.)

As has been shown by Hahn [43] (and then generalized by Neumann [44]) *the set* $\mathbb{K}((t^G))$ *together with the addition and multiplication defined by equation* (3.1) *forms a field.*

Recall that $(G, +)$ is called a *root group* if for any integer n and every $g \in G$ there exists $g' \in G$ such that $ng' = g$. We need also the notion of *universal field*. A field $\mathbb{K}$ is said to be *universal* if every other field $\mathbb{K}'$ of the same cardinal number and the same characteristic as $\mathbb{K}$ is isomorphic to some subfield of $\mathbb{K}$.

The following two important theorems have been proved by MacLane [45] (see also [41]):

Theorem 3.1. *If the coefficient field* $\mathbb{K}$ *is algebraically closed and the ordered Abelian group G is a root group, then the power series field* $\mathbb{K}((t^G))$ *is algebraically closed.*

Theorem 3.2. *If the coefficient field* $\mathbb{K}$ *is algebraically closed and the ordered Abelian group G is a root group and it contains an element different from the neutral element* 0*, then the power series field* $\mathbb{K}((t^G))$ *is universal.*

Suppose that the coefficient field $\mathbb{K}$ is formally real and is ordered by P. Then $\mathbb{K}((t^G))$ is a formally real field and there exists a natural order P' of $\mathbb{K}((t^G))$ generated by the order P. This order is defined as follows:

Definition 3.3. *If* $a = \sum_{g\in G} a_g t^g$, $a_g \in \mathbb{K}$ *and* g_0 *is the least element of supp a, then* $a > 0$ *iff* $a_{g_0} > 0$.

For the case when the coefficient field $\mathbb{K}$ is formally real one can rewrite theorem 3.1 in the form (see also Alling [46]).

Theorem 3.1′. *If the coefficient field* $\mathbb{K}$ *is real closed and the ordered Abelian group G is a root group, then the power series field* $\mathbb{K}((t^G))$ *is real closed.*

The fundamental object in the usual deformation quantization construction is an associative algebra $(C^\infty(M)((\hbar)), *)$ over the complex field $\mathbb{C}((\hbar)) = \mathbb{R}((\hbar)) + \sqrt{-1}\mathbb{R}((\hbar))$. We discuss this algebra in more detail in section 5. Here we note only that $C^\infty(M)((\hbar))$ denotes the set of formal power series on the group $\mathbb{Z}$ with coefficients being smooth complex functions on a symplectic manifold M. (As is used in deformation quantization the parameter t is denoted by $\hbar$.)

However, in the light of theorems 2.3, 3.1, 3.2 and 3.1′ it seems more convenient to deal with the algebra $(C^\infty(M)((\hbar^{\mathbb{Q}})), *)$ over the complex field $\mathbb{C}((\hbar^{\mathbb{Q}})) = \mathbb{R}((\hbar^{\mathbb{Q}})) + \sqrt{-1}\mathbb{R}((\hbar^{\mathbb{Q}}))$ where $(\mathbb{Q}, +)$ is the group of rational numbers. This conclusion can also be justified from the analytical point of view.

To this end define a valuation $\nu : \mathbb{C}((\hbar^{\mathbb{Q}})) \to \mathbb{R} \cup \{\infty\}$ (or $\mathbb{R}((\hbar^{\mathbb{Q}})) \to \mathbb{R} \cup \{\infty\}$) as follows:

$$\nu(a) = \min(\operatorname{supp} a) \qquad a \in \mathbb{C}((\hbar^{\mathbb{Q}})) \quad (\text{or } \mathbb{R}((\hbar^{\mathbb{Q}}))). \tag{3.2}$$

Then the metric $d_\nu : \mathbb{C}((\hbar^{\mathbb{Q}})) \times \mathbb{C}((\hbar^{\mathbb{Q}})) \to \mathbb{R}$ (or $\mathbb{R}((\hbar^{\mathbb{Q}})) \times \mathbb{R}((\hbar^{\mathbb{Q}})) \to \mathbb{R}$) is given by (2.1).

Analogously with what has been done in [25] (proposition 2) one can prove:

Proposition 3.1. $(\mathbb{C}((\hbar^{\mathbb{Q}})), d_\nu)$ *and* $(\mathbb{R}((\hbar^{\mathbb{Q}})), d_\nu)$ *are complete metric spaces.*

It is also a simple matter to show that the valuation (3.2) is compatible with the ordering of $\mathbb{R}((\hbar^{\mathbb{Q}}))$ given by definition 3.3, i.e. $\mathcal{T}_\nu = \mathcal{T}_0$, where the topologies $\mathcal{T}_\nu$ and $\mathcal{T}_0$ are defined in section 2.

Remark. Bordemann and Waldmann [25] deal with some subfields of $\mathbb{C}((\hbar^{\mathbb{Q}}))$ (or $\mathbb{R}((\hbar^{\mathbb{Q}}))$) defined as follows:

(1) The field of *formal Newton–Puiseux (NP) series*

$$\mathbb{C}\langle\langle\hbar^*\rangle\rangle := \{a \in \mathbb{C}((\hbar^{\mathbb{Q}})) : \exists\, N \in \mathbb{N}\; N \cdot \operatorname{supp} a \subset \mathbb{Z}\};$$

(2) The field of *formal completed Newton–Puiseux (CNP) series*

$$\mathbb{C}\langle\langle\hbar\rangle\rangle := \{a \in \mathbb{C}((\hbar^{\mathbb{Q}})) : \operatorname{supp} a \cap [p, q] \textit{ is finite for any } p, q \in \mathbb{Q}\},$$

and similarly for $\mathbb{R}\langle\langle\hbar\rangle\rangle$ and $\mathbb{R}\langle\langle\hbar^*\rangle\rangle$. In proposition 2 of [25] it is shown that $(\mathbb{C}\langle\langle\hbar\rangle\rangle, d_\nu)$ and $(\mathbb{R}\langle\langle\hbar\rangle\rangle, d_\nu)$ are complete metric spaces. Moreover, $(\mathbb{C}\langle\langle\hbar^*\rangle\rangle, d_\nu)$ (or $(\mathbb{R}\langle\langle\hbar^*\rangle\rangle, d_\nu)$) is dense in $(\mathbb{C}\langle\langle\hbar\rangle\rangle, d_\nu)$ (or $(\mathbb{R}\langle\langle\hbar\rangle\rangle, d_\nu)$). Then in theorem 1 of [25] it is proved that both fields, $\mathbb{C}\langle\langle\hbar^*\rangle\rangle$ and $\mathbb{C}\langle\langle\hbar\rangle\rangle$, are algebraically closed ($\mathbb{R}\langle\langle\hbar^*\rangle\rangle$ and $\mathbb{R}\langle\langle\hbar\rangle\rangle$ are real closed).

It is evident that $(\mathbb{C}\langle\langle\hbar\rangle\rangle, d_\nu)$ and, consequently, $(\mathbb{C}\langle\langle\hbar^*\rangle\rangle, d_\nu)$ are not dense metric spaces in $(\mathbb{C}((\hbar^{\mathbb{Q}})), d_\nu)$. However, since $(\mathbb{C}((\hbar^{\mathbb{Q}})), d_\nu)$ is complete and the field $\mathbb{C}((\hbar^{\mathbb{Q}}))$ is algebraically closed then $\mathbb{C}((\hbar^{\mathbb{Q}}))$ can be applied to the GNS construction in deformation quantization analogously as is in the case of $\mathbb{C}\langle\langle\hbar\rangle\rangle$ [25, 26].

4. Robertson and Hadamard–Robertson theorems for an arbitrary ordered field

The well-known Heisenberg uncertainty relation between two canonical observables admits several generalizations. One of them was given by Robertson [5] and Schrödinger [6]. These results were then generalized to an arbitrary number of observables by Robertson [7]. Recently a revival of interest in the important Robertson work can be observed ([12–17] and references given therein).

In this section we are going to generalize Robertson's results to an arbitrary formal real ordered field. Let $(\mathbb{K}, P)$ be a formally real ordered field and $\mathbb{K}^c := \mathbb{K}(\mathrm{i}) = \mathbb{K} + \mathrm{i}\mathbb{K}$, $\mathrm{i} \equiv \sqrt{-1}$, its complexification.

Let V be a vector space over $\mathbb{K}^c$.

Definition 4.1. *A Hermitian form on V is a map $\phi : V \times V \to \mathbb{K}^c$ satisfying the following properties.*

(i) $\phi(c_1 v_1 + c_2 v_2, w) = \overline{c_1}\phi(v_1, w) + \overline{c_2}\phi(v_2, w)$
(ii) $\phi(v, c_1 w_1 + c_2 w_2) = c_1\phi(v, w_1) + c_2\phi(v, w_2)$
(iii) $\overline{\phi(v, w)} = \phi(w, v)$

$\forall v_1, v_2, w_1, w_2, v, w \in V$, $\forall c_1, c_2 \in \mathbb{K}^c$.

In this paper the overbar denotes the complex conjugation.

(*Note*. A map $\psi : V \times V \to \mathbb{K}^c$ is said to be a *sesquilinear form* if it satisfies (i) and (ii) [32].)

Hermitian form $\phi : V \times V \to \mathbb{K}^c$ is said to be *positive definite* if $\phi(v, v) > 0$ *for all nonzero* $v \in V$; and it is said to be *non-negative definite* if $\phi(v, v) \geqslant 0$ $\forall v \in V$.

Suppose that $\dim V = n$. Let $(e_1, \ldots, e_n)$ be any basis of V and let $v = \sum_{j=1}^{n} v_j e_j$ be any vector of V. Then from definition 4.1 one gets

$$\phi(v, v) = \sum_{j,k=1}^{n} \phi_{jk}\overline{v_j}v_k \qquad \overline{\phi_{jk}} = \phi_{kj} \tag{4.1}$$

where $\phi_{jk} := \phi(e_j, e_k)$.

We can write $\phi_{jk} = a_{jk} + \mathrm{i}b_{jk}$, $a_{jk}, b_{jk} \in \mathbb{K}$. From (4.1) it follows that $a_{jk} = a_{kj}$ and $b_{jk} = -b_{kj}$. So the $n \times n$ matrix (ϕ_{jk}) over $\mathbb{K}^c$ is *Hermitian*, the matrix (a_{jk}) over $\mathbb{K}$ is *symmetric* and the matrix (b_{jk}) over $\mathbb{K}$ is *skew-symmetric*.

Now we are in a position to prove a generalization of the Robertson theorem to an arbitrary formally real ordered field.

Theorem 4.1 (Robertson)**.** *With the notation as above, let $\phi : V \times V \to \mathbb{K}^c$ be a non-negative definite Hermitian form on V. Then* $\det(a_{jk}) \geqslant \det(b_{jk})$. *If ϕ is positive definite then* $\det(a_{jk}) > \det(b_{jk})$. *If* $\det(a_{jk}) = 0$ *then* $\det(b_{jk}) = 0$.

Proof. Let $v = \sum_{j=1}^{n} v_j e_j$ be any vector in V. Write $\mathbb{K}^c \ni v_j = x_j + \mathrm{i}y_j$, $x_j, y_j \in \mathbb{K}$. Then $\phi(v, v) = \sum_{j,k=1}^{n} a_{jk}(x_j x_k + y_j y_k) - 2\sum_{j,k=1}^{n} b_{jk} x_j y_k$. Letting $y_j = 0$, one quickly finds that

$$\phi(v, v) \geqslant 0 \qquad \forall\, v \in V \quad \Rightarrow \quad \sum_{j,k=1}^{n} a_{jk} x_j x_k \geqslant 0 \qquad \forall x_j \in \mathbb{K}.$$

Consequently, $\sum_{j,k=1}^{n} a_{jk} x_j x_k$ is a non-negative definite quadratic form and in particular it follows that $\det(a_{jk}) \geqslant 0$.

The $n \times n$ matrix (b_{jk}) is skew-symmetric. Hence if n is an odd number then $\det(b_{jk}) = 0$ and the theorem holds. Thus we assume that n is an even number $n = 2m$.

The proof is divided into two parts.

(1) $\det(a_{jk}) > 0$. Here we follow Robertson [7] (see also [16]).

The matrices (a_{jk}) and $(c_{jk}) := \mathrm{i}(b_{jk})$ are Hermitian. One can find a $2m \times 2m$ matrix D over $\mathbb{K}$, with $\det(D) \neq 0$, such that the transformed matrix $(a'_{jk}) := D^T(a_{jk})D$, where D^T denotes the transposed matrix of D, is a diagonal matrix with all its diagonal elements positive. According to theorem 2.4 we can extend the ordered field $(\mathbb{K}, P)$ to its closure $(\mathbb{K}', P')$. So without any loss of generality we assume from the very beginning that $(\mathbb{K}, P)$ *is real closed*. With this assumption and by theorem 2.3 the matrix D over $\mathbb{K}$ can be found such that the matrix (a'_{jk}) is the unit matrix 1. It is obvious that the matrix $(c'_{jk}) := D^T(c_{jk})D$ is still Hermitian. Therefore, there exists a $2m \times 2m$ unitary matrix U over $\mathbb{K}^c$ ($U^\dagger U = 1$, $U^\dagger := \overline{U}^T$) such that the transformed matrix $(c''_{jk}) := (DU)^{\dagger}(c_{jk})(DU)$ is diagonal. Moreover, the transformed matrix $(a''_{jk}) := U^{\dagger}(a'_{jk})U = 1$. Hence, finally we get

$$(c''_{jk}) = (DU)^{\dagger}(c_{jk})(DU) = \mathrm{diag}(\lambda_1, \ldots, \lambda_{2m}) \qquad (a''_{jk}) = (DU)^{\dagger}(a_{jk})(DU) = 1 \tag{4.2}$$

where $\lambda_1, \ldots, \lambda_{2m} \in \mathbb{K}$ are the solutions of the characteristic equation

$$\det((c'_{jk}) - \lambda 1) = 0. \tag{4.3}$$

Since the matrix (c'_{jk}) is skew-symmetric then if λ is a solution of the characteristic equation (4.3) then $-\lambda$ is also a solution of this equation.

Therefore, the matrix (c''_{jk}) is of the form

$$(c''_{jk}) = \mathrm{diag}(\lambda_1, -\lambda_1, \ldots, \lambda_m, -\lambda_m) \qquad \lambda_1, \ldots, \lambda_m \in \mathbb{K}. \tag{4.4}$$

By equations (4.2) and (4.4) one quickly finds that the transformed matrix (ϕ''_{jk}) of the Hermitian form ϕ reads

$$(\phi''_{jk}) := (DU)^{\dagger}(\phi_{jk})(DU) = (a''_{jk}) + (c''_{jk}) = \mathrm{diag}(1 + \lambda_1, 1 - \lambda_1, \ldots, 1 + \lambda_m, 1 - \lambda_m). \tag{4.5}$$

Since ϕ is a non-negative definite Hermitian form then

$$1 \pm \lambda_k \geqslant 0 \qquad k = 1, \ldots, m. \tag{4.6}$$

From (4.2) and (4.4) one gets

$$\det(a_{jk}) = (\det D)^{-2} \qquad \det(c_{jk}) = (\det D)^{-2}(-1)^m \lambda_1^2 \cdots \lambda_m^2. \tag{4.7}$$

But $\det(c_{jk}) = \det(\mathrm{i}b_{jk}) = \mathrm{i}^{2m} \det(b_{jk}) = (-1)^m \det(b_{jk})$.

Substituting this relation into (4.7), employing also the fact that by (4.6) $\lambda_1^2 \cdots \lambda_m^2 \leqslant 1$ we obtain that the inequality $\det(a_{jk}) \geqslant \det(b_{jk})$ holds true.

Observe that if ϕ is a positive definite Hermitian form then $\det(\phi_{jk}) > 0 \Rightarrow \det(a_{jk}) > 0$. Moreover, in (4.6) one has the strict inequalities $1 \pm \lambda_k > 0$ and consequently, we obtain the strict inequality $\det(a_{jk}) > \det(b_{jk})$.

This completes the first part of the proof. Consider now the second part when:

(2) $\det(a_{jk}) = 0$. Then it follows that also $\det(\phi_{jk}) = 0$.

There exists an unitary matrix U over $\mathbb{K}^c$ such that

$$(\phi'_{jk}) := U^{\dagger}(\phi_{jk})U = \mathrm{diag}(\phi_1, \ldots, \phi_q, 0, \ldots, 0) \qquad q < 2m \qquad \phi_1, \ldots, \phi_q > 0. \tag{4.8}$$

Define now

$$(\phi'_{jk}(x)) := \mathrm{diag}(\phi_1, \ldots, \phi_q, x, \ldots, x) \qquad x \geqslant 0. \tag{4.9}$$

It is evident that the Hermitian form $\phi(x)$ given by the matrix

$$(\phi_{jk}(x)) = U(\phi'_{jk}(x))U^{\dagger} \qquad x \geqslant 0 \tag{4.10}$$

is positive definite for every $x > 0$. Moreover, $(\phi_{jk}(0)) = (\phi_{jk})$, i.e., $\phi(0) = \phi$.

We split $(\phi_{jk}(x))$ as before

$$\begin{aligned} &\phi_{jk}(x) = a_{jk}(x) + \mathrm{i}b_{jk}(x) \qquad x \geqslant 0 \qquad a_{jk}(x), b_{jk}(x) \in \mathbb{K} \\ &a_{jk}(x) = a_{kj}(x) \qquad b_{jk}(x) = -b_{kj}(x) \qquad a_{jk}(0) = a_{jk} \qquad b_{jk}(0) = b_{jk}. \end{aligned} \tag{4.11}$$

Since $\det(\phi_{jk}(x)) > 0\ \forall x > 0$ then also $\det(a_{jk}(x)) > 0\ \forall x > 0$ and by the first part (1) of the proof one has

$$\det(a_{jk}(x)) > \det(b_{jk}(x)) \qquad \forall x > 0. \tag{4.12}$$

From (4.8), (4.9), (4.10), (4.11) and the fact that $\det(a_{jk}) = 0$ it follows that

$$\begin{aligned} \det(a_{jk}(x)) &= \det(a_{jk}) + \sum_{l=1}^{r\leqslant 2m} d_l x^l = \sum_{l=1}^{r\leqslant 2m} d_l x^l \\ \det(b_{jk}(x)) &= \det(b_{jk}) + \sum_{l=1}^{s\leqslant 2m} f_l x^l \qquad d_l,\ f_l \in \mathbb{K} \end{aligned} \tag{4.13}$$

Consequently, by (4.12) and (4.13)

$$\begin{aligned} &\sum_{l=1}^{p\leqslant 2m} g_l x^l - \det(b_{jk}) > 0 \qquad \forall\, x > 0 \\ &\sum_{l=1}^{p\leqslant 2m} g_l x^l := \sum_{l=1}^{r\leqslant 2m} d_l x^l - \sum_{l=1}^{s\leqslant 2m} f_l x^l. \end{aligned} \tag{4.14}$$

Since (b_{jk}) is a skew-symmetric matrix over the formally real field $\mathbb{K}$ then $\det(b_{jk}) \geqslant 0$. Hence, $g_l \neq 0$ for some l.

We will show that $\det(b_{jk}) = 0$.

Suppose that $\det(b_{jk}) > 0$. The inequality (4.14) yields

$$\left(\sum_{l=1}^{p\leqslant 2m} |g_l| x^l - \det(b_{jk}) \right) > 0 \qquad \forall x > 0 \tag{4.15}$$

Without any loss of generality one can assume that all $g_l \neq 0$. Put then

$$x = \min\left(\frac{\det(b_{jk})}{2p|g_1|}, \sqrt{\frac{\det(b_{jk})}{2p|g_2|}}, \ldots, \sqrt[p]{\frac{\det(b_{jk})}{2p|g_p|}} \right). \tag{4.16}$$

Remember that, as has been pointed out in the first part (1) of our proof, without any loss of generality one can consider $\mathbb{K}$ to be a real closed field. So (4.16) is well defined by theorem 2.3.

Substituting x given by (4.16) into (4.15) we infer that

$$\left(\frac{\det(b_{jk})}{2} - \det(b_{jk}) \right) > 0 \quad \Rightarrow \quad \det(b_{jk}) < 0 \tag{4.17}$$

This contradicts the assumption: $\det(b_{jk}) > 0$. Consequently, $\det(b_{jk}) = 0$ and the proof is complete. □

Remark. Note that using analytical methods a different proof of the second part (2) of theorem 4.1 can be given. Namely, taking the limit of both sides of the inequality (4.14) when $x \to 0^+$ one immediately gets

$$\lim_{x\to 0^+} \left(\sum_{l=1}^{p\leqslant 2m} g_l x^l - \det(b_{jk}) \right) \geqslant 0. \tag{4.18}$$

As $\lim_{x\to 0^+} \left(\sum_{l=1}^{p\leqslant 2m} g_l x^l \right) = 0$ and $\det(b_{jk}) \geqslant 0$ we obtain $\det(b_{jk}) = 0$.

From the proof of theorem 4.1 (especially see (4.5) and (4.7)) we find that for $n = 2$, i.e. $m = 1$, the following corollary holds:

Corollary 4.1. *If* $n = 2$ *then* $\det(a_{jk}) = \det(b_{jk})$ *iff* $\det(\phi_{jk}) = 0$.

One can prove a useful lemma which will be employed to generalize the Hadamard–Robertson theorem.

Keeping the notation as above one has

Lemma 4.1. *Let $\phi : V \times V \to \mathbb{K}^c$ be a non-negative definite Hermitian form on V. Then* $\det(a_{jk}) \geqslant \det(\phi_{jk})$. *Equality* $\det(a_{jk}) = \det(\phi_{jk})$ *holds iff* $\det(a_{jk}) = 0$ *or* $(\phi_{jk}) = (a_{jk})$.

Proof. As before $\phi_{jk} = a_{jk} + \mathrm{i}b_{jk}$, where $a_{jk} = a_{kj}$ and $b_{jk} = -b_{kj}$ are elements of $\mathbb{K}$. We put $c_{jk} := \mathrm{i}b_{jk}$.

If $\det(a_{jk}) = 0$ then $\det(\phi_{jk}) = 0$ and the lemma is valid.

Let $\det(a_{jk}) > 0$. Suppose $\dim V = n$. Analogously as in theorem 4.1 one can choose an $n \times n$ matrix D over $\mathbb{K}$ such that $(a'_{jk}) := D^T(a_{jk})D = 1$. Obviously the matrix $(c'_{jk}) := D^T(c_{jk})D$ is Hermitian and skew-symmetric. Then a unitary $n \times n$ matrix can be found for which

$$(c''_{jk}) := U^\dagger(c'_{jk})U = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \qquad \text{and} \qquad (a''_{jk}) := U^\dagger(a'_{jk})U = 1$$

where $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ are the solutions of the characteristic equation (4.3). The $n \times n$ matrix (c'_{jk}) is skew-symmetric and as before it follows that if λ is a solution of the characteristic equation then $-\lambda$ is also a solution.

Hence in the case of even n, $n = 2m$, we have $(c''_{jk}) = \mathrm{diag}(\lambda_1, -\lambda_1, \ldots, \lambda_m, -\lambda_m)$, and in the case when n is odd, $n = 2m + 1$, the matrix $(c''_{jk}) = \mathrm{diag}(\lambda_1, -\lambda_1, \ldots, \lambda_m, -\lambda_m, 0)$. Consequently, $(\phi''_{jk}) = \mathrm{diag}(1 + \lambda_1, 1 - \lambda_1, \ldots, 1 + \lambda_m, 1 - \lambda_m)$ for $n = 2m$, and $(\phi''_{jk}) = \mathrm{diag}(1 + \lambda_1, 1 - \lambda_1, \ldots, 1 + \lambda_m, 1 - \lambda_m, 1)$ for $n = 2m + 1$.

Since ϕ is non-negative definite then $1 \pm \lambda_k \geqslant 0$ for all k. Therefore, $\det(\phi''_{jk}) = \left(1 - \lambda_1{}^2\right) \cdots \left(1 - \lambda_m{}^2\right) \leqslant 1$. So $\det(\phi''_{jk}) \leqslant \det(a''_{jk})$, and the equality $\det(\phi''_{jk}) = \det(a''_{jk})$ holds iff $\lambda_1 = \cdots = \lambda_m = 0$, i.e. iff $(c''_{jk}) = 0$. This yields $\det(\phi_{jk}) \leqslant \det(a_{jk})$ and the equality $\det(\phi_{jk}) = \det(a_{jk})$ holds iff $(c_{jk}) = 0$. The proof is complete. □

To obtain a generalization of the Heisenberg uncertainty principle to any formally real ordered field it is necessary to generalize first the Hadamard–Robertson theorem [7].

Theorem 4.2 (Hadamard–Robertson). *Let $\phi : V \times V \to \mathbb{K}^c$ be a non-negative definite Hermitian form on a vector space V of dimension n over $\mathbb{K}^c$. Then,*

(i) $\phi_{11} \ldots \phi_{nn} \geqslant \det(a_{jk}) \geqslant \det(\phi_{jk})$, $\phi_{11} \ldots \phi_{nn} \geqslant \det(a_{jk}) \geqslant \det(b_{jk})$

(ii) $\phi_{11} \ldots \phi_{nn} = \det(a_{jk}) = \det(\phi_{jk}) \Leftrightarrow \phi_{kk} = 0$ *for some k, or $(\phi_{jk}) = (a_{jk})$ is diagonal.*

(iii) $\phi_{11} \ldots \phi_{nn} = \det(b_{jk}) \Leftrightarrow \phi_{kk} = 0$ *for some k or (a_{jk}) is diagonal and* $\det(b_{jk}) = \det(a_{jk})$.

Proof. (i) From theorem 4.1 and lemma 4.1 one has: $\det(a_{jk}) \geqslant \det(b_{jk})$ and $\det(a_{jk}) \geqslant \det(\phi_{jk})$, respectively. Hence it remains only to prove that $\phi_{11} \cdots \phi_{nn} \geqslant \det(a_{jk})$. But as $\phi_{kk} = a_{kk}$ for $k = 1, \ldots, n$ this inequality is equivalent to

$$a_{11} \ldots a_{nn} \geqslant \det(a_{jk}). \tag{4.19}$$

From the assumption that the Hermitian form $\phi : V \times V \to \mathbb{K}^c$ is non-negative definite it follows that the quadratic form

$$\sum_{j,k=1}^{n} a_{jk} x_j x_k \qquad x_j \in \mathbb{K} \tag{4.20}$$

is also non-negative definite. Keeping this in mind we prove (4.19) by induction with respect to the dimension of V. For $\dim V = 1$ the inequality (4.19) holds trivially. Assume that (4.19) is valid for $\dim V = n-1, n \geqslant 2$. Let now $\dim V = n$. We can find an $n \times n$ orthogonal matrix R over $\mathbb{K}$ of the form

$$R = \begin{pmatrix} r_{11} & \dots & r_{1,n-1} & 0 \\ \cdot & \dots & \cdot & \cdot \\ r_{n-1,1} & \dots & r_{n-1,n-1} & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix} \qquad R^T R = 1 \tag{4.21}$$

such that

$$(a'_{jk}) := R^T (a_{jk}) R = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & a'_{1n} \\ 0 & \lambda_2 & \dots. & 0 & a'_{2n} \\ \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & \dots & \lambda_{n-1} & a'_{n-1,n} \\ a'_{1n} & a'_{2n} & \dots. & a'_{n-1,n} & a'_{nn} \end{pmatrix} \qquad \lambda_1, \dots, \lambda_{n-1} \geqslant 0$$

$$a'_{nn} = a_{nn}. \tag{4.22}$$

One quickly finds that

$$\det(a'_{jk}) = \det(a_{jk}) = \lambda_1 \cdots \lambda_{n-1} a_{nn} - (a'_{1n})^2 \lambda_2 \cdots \lambda_{n-1} - \lambda_1 (a'_{2n})^2 \lambda_3 \cdots \lambda_{n-1} - \cdots - \lambda_1 \lambda_2 \cdots \lambda_{n-2} (a'_{n-1,n})^2 \leqslant \det(A_{n-1}) a_{nn} \tag{4.23}$$

where A_{n-1} is the $(n-1) \times (n-1)$ matrix over $\mathbb{K}$ defined by

$$A_{n-1} := \begin{pmatrix} a_{11} & \dots & a_{1,n-1} \\ a_{21} & \dots & a_{2,n-1} \\ \cdot & \dots & \cdot \\ a_{n-1,n} & \dots & a_{n-1,n-1} \end{pmatrix} \qquad \det A_{n-1} = \lambda_1 \dots \lambda_{n-1}.$$

Since the quadratic form (4.20) is non-negative definite, then the quadratic form $\sum_{j,k=1}^{n-1} a_{jk} x_j x_k$, $x_j \in \mathbb{K}$, is also non-negative definite. Consequently, the inductive assumption gives

$$a_{11} \dots a_{n-1,n-1} \geqslant \det(A_{n-1}).$$

Substituting this into (4.23) one gets (4.19) and the proof of (i) is complete.

(ii) $\Leftarrow$ If $\phi_{kk} = 0$ for some k or $(\phi_{jk}) = (a_{jk})$ is diagonal then

$$\phi_{11} \cdots \phi_{nn} = \det(a_{jk}) = \det(\phi_{jk}). \tag{4.24}$$

$\Rightarrow$ Assume that (4.24) holds. Hence, from lemma 4.1 we conclude that

$$\det(\phi_{jk}) = \det(a_{jk}) = 0 \qquad \text{or} \qquad (\phi_{jk}) = (a_{jk}).$$

Obviously, $\det(\phi_{jk}) = 0$ with (4.24) implies that $\phi_{kk} = 0$ for some k. Suppose then that (4.24) is valid and $\det(\phi_{jk}) > 0$. Now $(a_{jk}) = (\phi_{jk})$ and in (4.22) $\lambda_1, \dots, \lambda_{n-1}, a_{nn} > 0$. So from (4.23) it follows that the equality $\det(a_{jk}) = \det(A_{n-1}) \cdot a_{nn}$ holds iff, $a'_{1n} = \cdots = a'_{n-1,n} = 0$.

This last condition by (4.21) and (4.22), is equivalent to $a_{1n} = \cdots = a_{n-1,n} = 0$.

Analogous considerations for A_{n-1} etc, lead to the conclusion that (4.24) with $\det(\phi_{jk}) > 0$ implies $(a_{jk}) = (\phi_{jk}) = \mathrm{diag}(\phi_{11}, \dots, \phi_{nn})$, $\phi_{kk} > 0$ for $k = 1, \dots, n$.

(iii) The proof is straightforward keeping in mind that $\det(b_{jk}) \geqslant 0$ and employing (i) and (ii). □

Finally, we would like to generalize to an arbitrary formally real ordered field $(\mathbb{K}, P)$ an interesting uncertainty relation for the trace of the matrix (ϕ_{jk}) (Trifonov [17]).

Proposition 4.1. *For any non-negative definite Hermitian form* $\phi : V \times V \to \mathbb{K}^c$*, the following inequality holds:*

$$\mathrm{Tr}(\phi_{jk}) \geqslant \frac{2}{n-1} \sum_{j<k}^{n} |b_{jk}| \tag{4.25}$$

for every n, where $n = \dim V$*. If n is even,* $n = 2m$*, then also*

$$\mathrm{Tr}(\phi_{jk}) \geqslant 2 \sum_{j=1}^{m} |b_{j,m+j}|. \tag{4.26}$$

Proof. Assume $j \neq k$. We start with the obvious relation

$$(a_{jj} + a_{kk})^2 \geqslant 4a_{jj}a_{kk}.$$

From the Hadamard–Robertson theorem 4.2 we have

$$a_{jj}a_{kk} \geqslant b_{jk}^2.$$

Consequently,

$$\begin{aligned} &a_{jj} + a_{kk} \geqslant 2|b_{jk}| \\ &a_{jj} + a_{kk} = 2|b_{jk}| \quad \Leftrightarrow \quad a_{jk} = 0 \qquad \text{and} \qquad a_{jj} = a_{kk} = |b_{jk}|. \end{aligned} \tag{4.27}$$

Using the relation

$$\mathrm{Tr}(\phi_{jk}) = \mathrm{Tr}(a_{jk}) = \frac{1}{n-1} \sum_{j<k}^{n} (a_{jj} + a_{kk})$$

and (4.27) one gets that (4.25) holds true.

If $n = 2m$ we can write

$$\mathrm{Tr}(\phi_{jk}) = \mathrm{Tr}(a_{jk}) = \sum_{j=1}^{m} (a_{jj} + a_{m+j,m+j}).$$

This with (4.27) gives (4.26) and the proposition is proved. □

5. Uncertainty relations in deformation quantization

Deformation quantization was introduced as an alternative approach to the description of quantum systems. In the fundamental work by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [19] it is suggested that quantization should be understood "... as a deformation of the structure of the algebra of classical observables, rather than a radical change in the nature of the observables". This construction is realized by a deformation of the usual product algebra of smooth functions on the phase space and then by a deformation of the Poisson algebra.

To be more precise: let (M, ω) be a symplectic manifold (ω denotes the symplectic form on M), and let $C^\infty(M)((\hbar))$ be the vector space over $\mathbb{C}((\hbar))$ of the formal power series

$$f = \sum_{k=-N}^{\infty} f_k(x)\hbar^k \tag{5.1}$$

where $f_k(x)$ are complex smooth functions on M, $f_k \in C^\infty(M)$.

Definition 5.1 [19, 25, 39]**.** *Deformation quantization on* (M, ω) *is an associative algebra* $(C^\infty(M)((\hbar)), *)$ *over the field* $\mathbb{C}((\hbar))$*, where the associative product* $*$*, called the star product, is given by*

$$f * g = \sum_{k=0}^{\infty} C_k(f, g)\hbar^k \qquad f, g \in C^\infty(M)((\hbar)) \tag{5.2}$$

with $C_k, k \geqslant 0$*, being bidifferential operators such that* $C_k(C^\infty(M) \times C^\infty(M)) \subset C^\infty(M)\ \forall k, C_k(1, f) = C_k(f, 1) = 0$ *for* $k \geqslant 1, C_0(f, g) = fg, C_1(f, g) - C_1(g, f) = \mathrm{i}\{f, g\}$ *and* $\{\cdot, \cdot\}$ *stands for the Poisson bracket.*

It has been proved [39, 47, 48] that deformation quantization exists on each symplectic manifold. Even more, recently Kontsevich [40] proved the existence of star product for an arbitrary Poisson manifold. Perhaps the most transparent construction of star product on an arbitrary symplectic manifold has been given by Fedosov [39] in terms of the geometry of the formal Weyl algebra bundles. For our purpose it is not necessary to consider Fedosov's construction in more detail.

As has been pointed out in section 3 it seems natural to extend the associative algebra $(C^\infty(M)((\hbar)), *)$ over the field $\mathbb{C}((\hbar))$ to $(C^\infty(M)((\hbar^{\mathbb{Q}})), *)$ over the field $\mathbb{C}((\hbar^{\mathbb{Q}}))$. In what follows we deal with such an extended deformation quantization.

To proceed further we need the definition of positive functionals and states in deformation quantization. These concepts are fundamental in the GNS construction developed by Bordemann *et al* [25, 26] and so seem basic to relate deformation quantization with quantum mechanics.

Analogously as in the theory of C^*-algebras one has [25, 49]:

Definition 5.2. *A* $\mathbb{C}((\hbar^{\mathbb{Q}}))$ *linear functional* ρ: $C^\infty(M)((\hbar^{\mathbb{Q}})) \to \mathbb{C}((\hbar^{\mathbb{Q}}))$ *is said to be positive if*

$$\rho(\overline{f} * f) \geqslant 0 \qquad \forall f \in C^\infty(M)((\hbar^{\mathbb{Q}})).$$

A positive linear functional ρ *is called a state if* $\rho(1) = 1$*.*

One can easily check that if a linear functional ρ is positive then

$$\overline{\rho(f * g)} = \rho(\overline{g} * \overline{f}) \tag{5.3}$$

and the *Cauchy–Schwarz inequality*

$$\rho(\overline{f} * g)\overline{\rho(\overline{f} * g)} \leqslant \rho(\overline{f} * f)\rho(\overline{g} * g) \tag{5.4}$$

holds true. In particular, taking in (5.3) $g = 1$ we get

$$\overline{\rho(f)} = \rho(\overline{f}). \tag{5.5}$$

Consequently, if $\overline{f} = f$ then $\rho(f) \in \mathbb{R}((\hbar^{\mathbb{Q}}))$.

From (5.3) and (5.5) it follows that

$$\rho(\overline{f * g} - \overline{g} * \overline{f}) = 0.$$

This condition is satisfied for any positive functional iff

$$\overline{f * g} = \overline{g} * \overline{f} \qquad \forall f, g \in C^\infty(M)((\hbar^{\mathbb{Q}})). \tag{5.6}$$

Note that it is always possible to construct a star product which satisfies (5.6) [39, 50].

Another fundamental concept in the GNS construction and employed in the present paper to describe intelligent states (section 6) is that of the *Gel'fand ideal*.

Definition 5.3. *Let $\rho : C^\infty(M)((\hbar^{\mathbb{Q}})) \to \mathbb{C}((\hbar^{\mathbb{Q}}))$ be a positive linear functional. Then the subspace $\mathcal{J}_\rho$ of $C^\infty(M)((\hbar^{\mathbb{Q}}))$*

$$\mathcal{J}_\rho := \{f \in C^\infty(M)((\hbar^{\mathbb{Q}})) : \rho(\overline{f} * f) = 0\}$$

is called the Gel'fand ideal of ρ.

It can be easily shown that by (5.3) and (5.4) $\mathcal{J}_\rho$ is a left ideal of $C^\infty(M)((\hbar^{\mathbb{Q}}))$, i.e. if $f \in \mathcal{J}_\rho$ then $g * f \in \mathcal{J}_\rho$ $\forall g \in C^\infty(M)((\hbar^{\mathbb{Q}}))$ and

$$\rho(\overline{f} * g) = 0 = \rho(g * f) \qquad \forall g \in C^\infty(M)((\hbar^{\mathbb{Q}})). \tag{5.7}$$

Let $\rho : C^\infty(M)((\hbar^{\mathbb{Q}})) \to \mathbb{C}((\hbar^{\mathbb{Q}}))$ be a positive linear functional. Define the sesquilinear form $\phi : C^\infty(M)((\hbar^{\mathbb{Q}})) \times C^\infty(M)((\hbar^{\mathbb{Q}})) \to \mathbb{C}((\hbar^{\mathbb{Q}}))$ by

$$\phi(f, g) := \rho(\overline{f} * g) \qquad f, g \in C^\infty(M)((\hbar^{\mathbb{Q}})). \tag{5.8}$$

(For the definition of a sesquilinear form see the note after definition 4.1.)

From (5.3) one quickly finds that

$$\overline{\phi(f, g)} = \phi(g, f). \tag{5.9}$$

It means that ϕ is a Hermitian form on $C^\infty(M)((\hbar^{\mathbb{Q}}))$. Moreover, since $\phi(f, f) = \rho(\overline{f} * f) \geqslant 0$ $\forall f \in C^\infty(M)((\hbar^{\mathbb{Q}}))$ then ϕ defined by (5.8) is a *non-negative definite Hermitian form*.

Now we are in a position to obtain uncertainty relations in deformation quantization. To this end, let $X_1, \ldots, X_n \in C^\infty(M)((\hbar^{\mathbb{Q}}))$ satisfy the reality conditions $\overline{X_j} = X_j$, $j = 1, \ldots, n$ (i.e. , X_j are *observables*) and let $\rho : C^\infty(M)((\hbar^{\mathbb{Q}})) \to \mathbb{C}((\hbar^{\mathbb{Q}}))$ be a state. Define *deviations from the mean* as follows:

$$\delta X_j := X_j - \rho(X_j). \tag{5.10}$$

Since $\overline{X_j} = X_j$ and ρ is a state then by (5.5) one gets

$$\overline{\delta X_j} = \delta X_j. \tag{5.11}$$

It is also evident that $\rho(\delta X_j) = 0$. Take

$$f := \sum_{j=1}^{n} v_j \delta X_j \qquad v_j \in \mathbb{C}((\hbar^{\mathbb{Q}})).$$

Then from (5.8) and (5.11) we have

$$\phi(f, f) = \rho\left(\sum_{j=1}^{n} \overline{v_j} \delta X_j * \sum_{k=1}^{n} v_k \delta X_k\right) = \sum_{j,k=1}^{n} \overline{v_j} v_k \rho(\delta X_j * \delta X_k) = \sum_{j,k=1}^{n} \overline{v_j} v_k \phi(\delta X_j, \delta X_k).$$

Define

$$\phi_{jk} := \rho(\delta X_j * \delta X_k) = \phi(\delta X_j, \delta X_k) \qquad \phi_{jk} \in \mathbb{C}((\hbar^{\mathbb{Q}})). \tag{5.12}$$

From (5.9) it follows that $\overline{\phi_{jk}} = \phi_{kj}$. Since $\phi(f, f) \geqslant 0$ then

$$\sum_{j,k=1}^{n} \phi_{jk} \overline{v_j} v_k \geqslant 0 \qquad \forall v_j \in \mathbb{C}((\hbar^{\mathbb{Q}})). \tag{5.13}$$

Consequently, the $n \times n$ Hermitian matrix (ϕ_{jk}) over $\mathbb{C}((\hbar^{\mathbb{Q}}))$ determines a non-negative Hermitian form (5.13).

We can now use the results of section 4.

First, as before we write $\phi_{jk} = a_{jk} + \mathrm{i}b_{jk}, a_{jk}, b_{jk} \in \mathbb{R}((\hbar^{\mathbb{Q}}))$. From (5.12) and (5.10) one gets

$$\begin{aligned} a_{jk} &= \tfrac{1}{2}\rho(\delta X_j * \delta X_k + \delta X_k * \delta X_j) = \tfrac{1}{2}\rho(X_j * X_k + X_k * X_j) - \rho(X_j)\rho(X_k) = a_{kj} \\ b_{jk} &= -\frac{\mathrm{i}}{2}\rho(\delta X_j * \delta X_k - \delta X_k * \delta X_j) = \frac{\hbar}{2}\rho(\{X_j, X_k\}_*) = -b_{kj} \end{aligned} \tag{5.14}$$

where $\{X_j, X_k\}_* := \frac{1}{\mathrm{i}\hbar}(X_j * X_k - X_k * X_j)$. In analogy with quantum mechanics and statistics the $n \times n$ symmetric matrix (a_{jk}) over $\mathbb{R}((\hbar^{\mathbb{Q}}))$ can be called the *dispersion* or *covariance matrix*. A diagonal element $a_{jj} = \rho(X_j * X_j) - (\rho(X_j))^2$ which we denote also by $(\Delta X_j)^2$ is the *variance* of X_j, and $\Delta X_j = \sqrt{a_{jj}}$ is the *uncertainty in* X_j (or *standard deviation of* X_j). The element a_{jk} for $j \neq k$ is the *covariance* of X_j *and* X_k.

Having all that theorem 4.1 leads to the following *Robertson–Schrödinger uncertainty relation* in deformation quantization:

$$\det\left(\frac{1}{2}\rho(\delta X_j * \delta X_k + \delta X_k * \delta X_j)\right) \geqslant \det\left(\frac{\hbar}{2}\rho(\{X_j, X_k\}_*)\right). \tag{5.15}$$

In particular for two observables X_1 and X_2 we get

$$\Delta X_1 \Delta X_2 \geqslant \tfrac{1}{2}\sqrt{(\hbar\rho(\{X_1, X_2\}_*))^2 + (\rho(X_1 * X_2 + X_2 * X_1) - 2\rho(X_1)\rho(X_2))^2}. \tag{5.16}$$

This is the deformation quantization analogue of the well known in quantum mechanics uncertainty relation given by Robertson [5] and Schrödinger [6]. Relation (5.16) has been found recently by Curtright and Zachos [29]. However, their result seems to be derived in the spirit of a strict deformation quantization which makes use of the Wigner function and not for the formal deformation quantization in the sense of Bayen *et al* [19] considered in the present paper.

Another uncertainty relation in deformation quantization which we call the *Heisenberg–Robertson uncertainty relation* follows immediately from the Hadamard–Robertson theorem (theorem 4.2), and it reads

$$(\Delta X_1)^2 \cdots (\Delta X_n)^2 \geqslant \det\left(\frac{\hbar}{2}\rho(\{X_j, X_k\}_*)\right). \tag{5.17}$$

Finally, employing proposition 4.1 one gets the *trace uncertainty relation*

$$(\Delta X_1)^2 + \cdots + (\Delta X_n)^2 \geqslant \frac{\hbar}{n-1}\sum_{j<k}^{n} |\rho(\{X_j, X_k\}_*)|. \tag{5.18}$$

6. Intelligent states in deformation quantization

In quantum mechanics the states that minimize the Heisenberg–Robertson or the Robertson–Schrödinger uncertainty relations play an important role in the theory of coherent and squeezed states and they are called *Heisenberg–Robertson* or *Robertson–Schrödinger intelligent states*, (*minimum uncertainty states, correlated coherent states*) [8–13, 16]. It seems reasonable to extend these notions to deformation quantization. Thus we have

Definition 6.1. *A state* $\rho : C^\infty(M)((\hbar^{\mathbb{Q}})) \to \mathbb{C}((\hbar^{\mathbb{Q}}))$ *is said to be a Heisenberg–Robertson intelligent state for* $X_1, \ldots, X_n$ *if*

$$(\Delta X_1)^2 \cdots (\Delta X_n)^2 = \det\left(\frac{\hbar}{2}\rho(\{X_j, X_k\}_*)\right). \tag{6.1}$$

If

$$\det\left(\frac{1}{2}\rho(\delta X_j * \delta X_k + \delta X_k * \delta X_j)\right) = \det\left(\frac{\hbar}{2}\rho(\{X_j, X_k\}_*)\right) \qquad (6.2)$$

then ρ *is called a Robertson–Schrödinger intelligent state for* $X_1, \ldots, X_n$.

From theorems 4.1 and 4.2 one can easily obtain that

$$(6.1) \Rightarrow (6.2).$$

Hence every Heisenberg–Robertson intelligent state is also a Robertson–Schrödinger intelligent state.

To have a deeper insight into the Robertson–Schrödinger intelligent states we prove some conditions under which (6.2) is satisfied.

Our results are the deformation quantization versions of the propositions found by Trifonov in the case of quantum mechanics (propositions: 1 and 3 of [12]).

Observe that by theorem 4.1 if $\det\left(\frac{1}{2}\rho(\delta X_j * \delta X_k + \delta X_k * \delta X_j)\right) = 0$ then also $\det\left(\frac{\hbar}{2}\rho(\{X_j, X_k\}_*)\right) = 0$. Hence, $\det\left(\frac{1}{2}\rho(\delta X_j * \delta X_k + \delta X_k * \delta X_j)\right) = 0$ is a sufficient condition for ρ to be a Robertson–Schrödinger intelligent state for $X_1, \ldots, X_n$. In the case when the number n of observables X_j is odd this condition is also necessary.

We can prove

Proposition 6.1. *Let* $\rho : C^\infty(M)((\hbar^{\mathbb{Q}})) \to \mathbb{C}((\hbar^{\mathbb{Q}}))$ *be a state and* $a_{jk} := \frac{1}{2}\rho(\delta X_j * \delta X_k + \delta X_k * \delta X_j)$, $j, k = 1, \ldots, n$. *Then* $\det(a_{jk}) = 0$ *iff there exist* $x_1, \ldots, x_n \in \mathbb{R}((\hbar^{\mathbb{Q}}))$ *such that* $\sum_{j=1}^n |x_j| > 0$ *and*

$$\rho\left(\sum_{j=1}^n x_j \delta X_j * \sum_{k=1}^n x_k \delta X_k\right) = 0 \qquad (6.3)$$

i.e., $\sum_{j=1}^n x_j \delta X_j$ *is an element of the Gel'fand ideal* $\mathcal{J}_\rho$ *of* ρ.

Proof (Compare with [12])**.** Assume that $\det(a_{jk}) = 0$. Then there exists an $n \times n$ orthogonal matrix $R = (r_{jk})$ over $\mathbb{R}((\hbar^{\mathbb{Q}}))$, $R^T R = 1$, such that

$$\begin{aligned} R^T(a_{jk})R &= R^T\left(\tfrac{1}{2}\rho(\delta X_j * \delta X_k + \delta X_k * \delta X_j)\right)R \\ &= \mathrm{diag}(\lambda_1, \ldots, \lambda_{q-1}, 0 \ldots, 0) \qquad 2 \leqslant q \leqslant n. \end{aligned}$$

Hence

$$\rho\left(\sum_{j=1}^n r_{jq} \delta X_j * \sum_{k=1}^n r_{kq} \delta X_k\right) = 0.$$

Denoting $x_j := r_{jq} \in \mathbb{R}((\hbar^{\mathbb{Q}}))$ one gets (6.3). This completes the first part of the proof.

Assume now that there exist $x_1, \ldots, x_n \in \mathbb{R}((\hbar^{\mathbb{Q}}))$ such that $\sum_{j=1}^n |x_j| > 0$ and (6.3) holds. Choose an $n \times n$ matrix $D = (d_{jk})$ over $\mathbb{R}((\hbar^{\mathbb{Q}}))$ such that $d_{j1} = x_j$, $j = 1, \ldots, n$, and $\det D \neq 0$.

Consider the transformed matrix $(a'_{jk}) = D^T(a_{jk})D$.

We have

$$a'_{1l} = \frac{1}{2}\rho\left(\sum_{j=1}^n x_j \delta X_j * \sum_{k=1}^n d_{kl} \delta X_k + \sum_{k=1}^n d_{kl} \delta X_k * \sum_{j=1}^n x_j \delta X_j\right) \qquad l = 1, \ldots, n.$$

Since $\sum_{j=1}^{n} x_j \delta X_j \in \mathcal{J}_\rho$ then by (5.7)

$$\rho\left(\sum_{j=1}^{n} x_j \delta X_j * g\right) = 0 = \rho\left(g * \sum_{j=1}^{n} x_j \delta X_j\right) \qquad \forall g \in C^\infty(M)((\hbar^{\mathbb{Q}})).$$

Therefore, $a'_{1l} = 0$ for $l = 1, \ldots, n$ and consequently, $\det(a'_{jk}) = 0$. But $\det(a'_{jk}) = (\det D)^2 \det(a_{jk})$ with $\det D \neq 0$. This yields $\det(a_{jk}) = 0$. The proof is complete. □

To find another sufficient condition that a given state ρ be a Robertson–Schrödinger intelligent state for $X_1, \ldots, X_n$ we deal with the case when n is an even number, $n = 2m$. Thus we have $X_1, \ldots, X_{2m} \in C^\infty(M)((\hbar^{\mathbb{Q}}))$ such that $\overline{X_j} = X_j$, $j = 1, \ldots, 2m$. Let δX_j be deviations from the mean as in (5.10). Introduce the following objects:

$$\begin{aligned} \delta A_\alpha &:= \tfrac{1}{2}(\delta X_\alpha + \mathrm{i}\delta X_{\alpha+m}) \\ \overline{\delta A_\alpha} &= \tfrac{1}{2}(\delta X_\alpha - \mathrm{i}\delta X_{\alpha+m}) \qquad \alpha = 1, \ldots, m. \end{aligned} \tag{6.4}$$

With all that one has

Proposition 6.2. *If there exists a linear transformation*

$$\begin{aligned} \delta A'_\alpha &= \sum_{\beta=1}^{m} (u_{\alpha\beta}\delta A_\beta + v_{\alpha\beta}\overline{\delta A_\beta}) \\ \overline{\delta A'_\alpha} &= \sum_{\beta=1}^{m} (\overline{v_{\alpha\beta}}\delta A_\beta + \overline{u_{\alpha\beta}}\,\overline{\delta A_\beta}) \qquad u_{\alpha\beta}, v_{\alpha\beta} \in \mathbb{C}((\hbar^{\mathbb{Q}})) \qquad \alpha, \beta = 1, \ldots, m \end{aligned} \tag{6.5}$$

such that

$$\det\begin{pmatrix} (u_{\alpha\beta}) & (v_{\alpha\beta}) \\ (\overline{v_{\alpha\beta}}) & (\overline{u_{\alpha\beta}}) \end{pmatrix} \neq 0 \tag{6.6}$$

and

$$\rho(\overline{\delta A'_\alpha} * \delta A'_\alpha) = 0 \qquad \alpha = 1, \ldots, m \tag{6.7}$$

($\delta A'_\alpha$ belongs to the Gel'fand ideal $\mathcal{J}_\rho$), then (6.2) is satisfied, i.e. ρ is a Robertson–Schrödinger intelligent state for $X_1, \ldots, X_{2m}$.

Proof (Compare with [12]). Following (6.4) define

$$\begin{aligned} \delta X'_\alpha &:= (\delta A'_\alpha + \overline{\delta A'_\alpha}) \\ \delta X'_{\alpha+m} &:= -\mathrm{i}(\delta A'_\alpha - \overline{\delta A'_\alpha}) \qquad \alpha = 1, \ldots, m. \end{aligned}$$

Obviously $\overline{\delta X'_j} = \delta X'_j$, $j = 1, \ldots, 2m$ and one can easily check that

$$\delta X'_j = \sum_{k=1}^{2m} d_{jk}\delta X_k \tag{6.8}$$

where under (6.4), (6.5) and (6.6) the $2m \times 2m$ matrix (d_{jk}) over $\mathbb{R}((\hbar^{\mathbb{Q}}))$ is non-singular, $\det(d_{jk}) \neq 0$. Straightforward calculations under the assumption (6.7) lead to the relation

$$\det\left(\frac{1}{2}\rho(\delta X'_j * \delta X'_k + \delta X'_k * \delta X'_j)\right) = \det\left(\frac{\hbar}{2}\rho(\{X'_j, X'_k\}_*)\right).$$

Consequently, by (6.8) equation (6.2) holds true. □

Employing corollary 4.1 for the case of two observables one can easily prove the next proposition.

Proposition 6.3. *A state* ρ *is a Robertson–Schrödinger intelligent state for* X_1, X_2 *iff there exist* $u_1, u_2 \in \mathbb{C}((\hbar^{\mathbb{Q}}))$ *such that* $u_1 \delta X_1 + u_2 \delta X_2 \in \mathcal{J}_\rho$.

Robertson–Schrödinger intelligent states for two observables in terms of Moyal star product and Wigner functions have been considered in [24, 29].

7. Concluding remarks

In this paper we have obtained uncertainty relations in deformation quantization formalism. To achieve this, first it was necessary to study a general theory of formal real ordered fields and to apply it to the case of formal power series. Having done all that we were able to generalize the Robertson and Hadamard–Robertson theorems to be valid for an arbitrary ordered field. This allowed us to formulate several uncertainty relations and to introduce the concept of intelligent states in deformation quantization. Of course further investigations in this direction are needed. In particular, one should consider some concrete set of observables and get examples of the corresponding intelligent states.

It is expected that the results of the present paper will give a better understanding of the relations between quantum mechanics and deformation quantization.

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